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THE SYNTHESIS OF INERTIAL CONTROLS FOR NON-STATIONARY SYSTEMS†

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The problem of an admissible synthesis of inertia controls for non-stationary systems with a multidimensional control with geometrical constraints on the control and its derivatives is considered. The problem is solved analytically for a linear system: a constructive structure of a family of controls is given, each of which solves the problem, the time of motion from the initial point at zero is calculated and the corresponding trajectory is found. For a non-linear system the problem is solved to a first approximation in the case when there are constraints on the control and on its derivatives. © 2003 Elsevier Ltd. All rights reserved.

1. INTRODUCTION

We will consider the problem of an admissible synthesis of bounded inertial controls for the system

$$\dot{x} = f(t, x, u), \quad x \in R^n, \quad u \in R^r, \quad t \in [t_0, t_1] \tag{1.1}$$

i.e. the problem of constructing a control $u = u(t, x)$, which transfers an arbitrary initial point $x(t_0) = x_0$ from a certain neighbourhood $Q(t_0)$ of the origin of coordinates to a point $x_1 = 0$ along a trajectory $x(t) \in Q(t)$ of the system

$$\dot{x} = f(t, x, u(t, x)) \tag{1.2}$$

in a finite time $T(t_0, x_0) \le t_1 - t_0$ and which satisfies, together with the derivatives $u^{(1)}(t, x), \ldots, u^{(l)}(t, x)$, by virtue of system (1.2), the constraints

$$\|u^{(k)}(t, x)\| \le d_k, \quad k = 0, 1, \ldots, l, \quad x \in Q(t), \quad t \in [t_0, t_0 + T) \tag{1.3}$$

where $d_0, \ldots, d_1$ are specified numbers.

Controls with such constraints were considered previously in [1] and were called inertial controls. Sets of controllability for linear systems with inertial controls were considered in [2, 3].

One arrives in a natural way at the problem of synthesizing an admissible control from the problem of the optimal synthesis of a control [1, 4–6], by dropping a certain quality criterion from the optimization.

The problem is solved in the same phase space, since, when the phase space is extended by introducing a new control $\upsilon = \dot{u}$ this approach gives a solution in the form $\upsilon = \upsilon(t, x, u)$, while it is necessary to obtain a control in the form $u = u(t, x)$.

Developing the results obtained in previous papers [7, 8], we will consider the problem of the admissible synthesis of controls with constraints on the control and its derivatives (unlike [7]) in the case of a non-stationary system and multidimensional control (unlike [8]). We will use the controllability function method [9, 10], which is based on the construction of a controllability function $\Theta(t, x)(\Theta(t, x) > 0$ when $x \ne 0$ and $\Theta(t, 0) = 0$ for $t \in [t_0, t_1])$ and controls $u(t, x) = \bar{u}(t, x, \Theta(t, x))$, such that the following inequality is satisfied

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$$\Lambda(\Theta(t,x);u(t,x)) \le -\beta\Theta^{1-1/\alpha}(t,x)$$
$$\Lambda(\Theta(t,x);u(t,x)) \doteq \frac{\partial\Theta(t,x)}{\partial t} + \sum_{i=1}^{n}\frac{\partial\Theta(t,x)}{\partial x_i}f_i(t,x,u(t,x)) \tag{1.4}$$

for certain $\beta > 0$, $\alpha > 0$. Inequality (1.4) denotes that the control is chosen in such a way that the motion occurs in the direction in which the function $\Theta(t, x)$ decreases. Satisfaction of this inequality ensures that the trajectory is incident on the origin of coordinates after a finite time.

The case when one must obtain the time of motion $T(t_0, x_0)$ from an arbitrary point $x(t_0) = x_0$ to the point $x_1 = 0$ when constructing the synthesizing controls is of interest. The case when the controllability function is the time of motion occurs, for example, when one uses the equality $\Lambda(\Theta(t,x); u(t,x)) = -1$ instead of condition (1.4). If, moreover, the control $u(t,x)$ is such that

$$\min_{u\in\Omega}\Lambda(\Theta(t,x);u) = \Lambda(\Theta(t,x);u(t,x)) = -1 \tag{1.5}$$

then, putting $\omega(t, x) = -\Theta(t, x)$, we obtain the fundamental equation of the method of dynamic programming – Bellman's equation [4, 5] $\max_{u\in\Omega}\Lambda(\omega(t,x);u) = 1$ for the speed of response problem.

The choice of the control using Eq. (1.5) can be treated from the position of minimizing the function $\Theta(t,x)$ as follows: the control $u(t,x)$ is chosen in such a way that the angle between the direction of the most rapid decrease in this function and the direction of motion is a minimum. In the controllability function method this angle is not necessarily a minimum.

When inequality (1.4) has the form

$$\Lambda(\Theta(t,x);u(t,x)) \le -\beta\Theta(t,x)$$

the function $\Theta(t,x)$ is the Lyapunov function. This inequality indicates that, for small values of Θ, the angle between the direction of motion and the direction in which the function $\Theta(t,x)$ decreases is no less than in the controllability function method, since $\Theta(t,x) \le \Theta^{1-1/\alpha}(t,x)$ when $\alpha \ge 1$. Hence, the angle between the direction of motion and the direction in which the function $\Theta(t, x)$ decreases in the controllability function method is no less than the angle in the dynamic programming method and no greater than in the Lyapunov function method.

It is of interest to construct vector controllability functions by analogy with the Lyapunov vector functions introduced by Bellman [11] and Matrosov [12]. Such functions were constructed previously in [9] for autonomous linear systems.

2. SOLUTION OF THE PROBLEM OF SYNTHESIZING INERTIAL CONTROLS FOR A LINEAR COMPLETELY CONTROLLABLE SYSTEM

Consider the linear system

$$\dot{x} = A(t)x + B(t)u, \quad x\in R^n, \quad u\in R^r; \quad A(t)\in C^{2n-2+l}, \quad B(t)\in C^{2n-1+l} \tag{2.1}$$

Here and everywhere henceforth, unless otherwise stated, we will assume that $t \in [t_0, t_1]$. Without loss of generality, we will also assume that $\operatorname{rank}B(t) = r$. We will put $\Delta = A(t) - Ed/dt$ (E is the identity matrix) and assume that

$$\operatorname{rank}(B(t), \Delta B(t), \dots, \Delta^{n-1}B(t)) = n \tag{2.2}$$

and this rank is realized in the column vectors of the matrix

$$K(t) = (b_1(t), \dots, \Delta^{n_1-1}b_1(t), \dots, b_r(t), \dots, \Delta^{n_r-1}b_r(t)); \quad n_1 + \dots + n_r = n \tag{2.3}$$

where $b_i(t)$ is the i-th column of the matrix $B(t)$. We will put

$$n_0 = \max_{1\le i\le r} n_i, \quad s_0 = 0, \quad s_k = n_1 + \dots + n_k, \quad k = 1, \dots, r$$

Suppose we have the expansions

$$\Delta^{n_i} b_i(t) = \sum_{j=1}^{r} \sum_{k=0}^{n_j-1} \gamma_{jk}^i(t) \Delta^k b_j(t), \quad i = 1, \ldots, r \tag{2.4}$$

where $\gamma_{jk}^i(t) \in C^n : \gamma_{jk}^i(t) = 0$ for $j < i, k > \min\{n_i, n_j - 1\}$ or $j \geq i, k > \min\{n_i - 1, n_j - 1\}$.

Following the approach proposed previously in [7], we choose a vector function $c_1(t), \ldots, c_r(t) \in C^n$ from the conditions

$$K^*(t) c_k(t) = e_{s_k}; \quad e_{s_k} = (0, \ldots, 0, 1, 0, \ldots, 0)^*, \quad k = 1, \ldots, r$$

(the asterisk denotes transposition).

Consider the non-degenerate matrix

$$L(t) = (c_1(t), \ldots, \Delta_*^{n_1-1} c_1(t), \ldots, c_r(t), \ldots, \Delta_*^{n_r-1} c_r(t))^*; \quad \Delta_* = A^*(t) + Ed/dt$$

We introduce the matrices

$$D(\Theta) = \operatorname{diag}(D_1(\Theta), \ldots, D_r(\Theta)), \quad D_i(\Theta) = \operatorname{diag}(\Theta^{-(n_i-k)/\alpha - 1/(2\alpha)})_{k=1}^{n_i}$$

$$H^\alpha = \operatorname{diag}(H_1^\alpha, \ldots, H_r^\alpha), \quad H_i^\alpha = \operatorname{diag}(-(n_i - k)/\alpha - 1/(2\alpha))_{k=1}^{n_i}, \quad i = 1, \ldots, r$$

Consider $\{F_\alpha^{-1}(\Theta\}_{\alpha \geq 1}$ – a family of positive-definite matrices of the form

$$F_\alpha^{-1}(\Theta) = \int_0^{\alpha\Theta^{1/\alpha}} \left(1 - \frac{t}{\alpha\Theta^{1/\alpha}}\right)^\alpha \exp(-A_0 t) B_0 B_0^* \exp(-A_0^*) dt \tag{2.5}$$

where the $n \times n$ matrix A_0 has the form $A_0 = \operatorname{diag}(A_{01}, \ldots, A_{0r})$, A_{0i} is an $n_i \times n_i$ matrix, the elements of the first subdiagonal of which are unity, and all the remaining elements are zeros, and B_0 is an $n \times r$ matrix in which the elements $(B_0)_{s_i i} = 1$ $(i = 1, \ldots, r)$, while all the remaining elements are equal to zero. The matrix $F_\alpha(\Theta)$ can be represented in the form [8]

$$F_\alpha(\Theta) = D(\Theta) F_\alpha D(\Theta) \tag{2.6}$$

The matrix $F_\alpha \equiv F_\alpha(1)$ satisfies the equality

$$F_\alpha A_1 + A_1^* F_\alpha = -F_\alpha + F_\alpha H^\alpha + H^\alpha F_\alpha \equiv -F^\alpha; \quad A_1 = A_0 - 1/2 B_0 B_0^* F_\alpha \tag{2.7}$$

Analytical inversion of matrices of the form (2.5) were carried out previously in [13].

Suppose $a_0 > 0$ is a so far arbitrary number. For $\alpha \geq 1$ we will determine the controllability function $\Theta_\alpha(t, x)$ when $x \neq 0$ from the equation

$$2a_0\Theta = (L^*(t) F_\alpha(\Theta) L(t) x, x) \tag{2.8}$$

and put

$$\Theta_\alpha(t, 0) = 0 \tag{2.9}$$

It is easy to show that the following assertion holds.

Assertion 1. For each $\alpha \geq 1$ Eqs (2.8) and (2.9) define a non-negative function $\Theta = \Theta_\alpha(t, x)$, continuous for all x and continuously differentiable for $x \neq 0$.

Suppose $\bar{\Theta} > 0$ is a certain number. We put

$$R_\alpha = \delta\sqrt{2a_0\bar{\Theta}/(L_{\max}^2 \|F_\alpha(\bar{\Theta})\|)}, \quad \delta \in (0, 1), \quad L_{\max} = \max_{t_0 \leq t \leq t_1} \|L(t)\|$$

Assertion 2. For each $\alpha \geq 1$, a positive number $c_\alpha \leq ((t_1 - t_0)/\alpha)^\alpha$ exists such that the set

$$Q_\alpha(t) = \{x : \Theta_\alpha(t, x) \leq c_\alpha\} \tag{2.10}$$

is bounded and $Q_\alpha(t) \subset Q_\alpha^1 \doteq \{x : \|x\| < R_\alpha\}$.

Proof. From the relations

$$\{x : (L^*(t)F_\alpha(\bar{\Theta})L(t)x, x) < 2a_0\bar{\Theta}\} \supset \{x : \|x\|^2 < 2a_0\bar{\Theta}/(L^2_{\max}\|F_\alpha(\bar{\Theta})\|)\}$$

we have

$$2a_0\bar{\Theta} > (L^*(t)F_\alpha(\bar{\Theta})L(t)x, x), \quad x \in Q^1_\alpha\backslash\{0\}$$

Since $(L^*(t)F_\alpha(\Theta)L(t)x, x)$ is a decreasing function of Θ, on the basis of the inequality

$$(L^*(t)F_\alpha(\Theta)L(t)x, x) \geq \|x\|^2/(L_0^2\|F_\alpha^{-1}(\Theta)\|); \quad L_0 = \max_{t_0 \leq t \leq t_1} \|L^{-1}(t)\|$$

we have

$$Q^1_\alpha \supset \{x \colon \Theta_\alpha(t, x) \leq R^2_\alpha/(2a_0L_0^2\|F_\alpha^{-1}(\bar{\Theta})\|)\}$$

Using the expression for R_α, we can therefore conclude that for

$$c_\alpha = \min\left\{\frac{\sigma\delta^2\bar{\Theta}}{L^2_{\max}L_0^2\|F_\alpha(\bar{\Theta})\|\|F_\alpha^{-1}(\bar{\Theta})\|}, \left(\frac{t_1 - t_0}{\alpha}\right)^\alpha\right\}, \quad \sigma \in (0, 1) \tag{2.11}$$

the inclusion $Q_\alpha(t) \subset Q^1_\alpha$ holds.

We specify the control $u^\alpha(t, x)$ for $x \in Q^1_\alpha(t)\backslash\{0\}$ by the formula

$$u^\alpha(t, x) = -M^{-1}(t)B_0^*(1/2F_\alpha(\Theta_\alpha(t, x))L(t) + \dot{L}(t) + L(t)A(t))x \tag{2.12}$$

where $M(t)$ is an upper-triangular $r \times r$ matrix with elements

$$m_{ii}(t) = 1, \quad m_{ij}(t) = (\Delta_*^{n_i - 1}c_i(t))^*b_j(t) \quad \text{for} \quad j > i, \quad i = 1, \ldots, r$$

Assertion 3. The derivative of the function $\Theta_\alpha(t, x)$, by virtue of system (2.1) with control $u^\alpha(t, x)$ of the form (2.12), satisfies the equality

$$\dot{\Theta}_\alpha(t, x) = -\Theta_\alpha^{1 - 1/\alpha}(t, x) \tag{2.13}$$

Proof. We will further assume $\Theta_\alpha = \Theta_\alpha(t, x)$. We put

$$y(\Theta, t, x) = D(\Theta)L(t)x, \quad P_0 = -1/2B_0^*F_\alpha, \quad \tilde{A}(t) = (\dot{L}(t) + L(t)A(t))L^{-1}(t)$$

Then Eq. (2.8) and control (2.12), by virtue of (2.6), take the form

$$2a_0\Theta_\alpha = (F_\alpha y(\Theta_\alpha, t, x), y(\Theta_\alpha, t, x)) \tag{2.14}$$

$$u^\alpha(t, x) = M^{-1}(t)(\Theta_\alpha^{-1/(2\alpha)}P_0y(\Theta_\alpha, t, x) - B_0^*\tilde{A}(t)L(t)x) \tag{2.15}$$

We will calculate the derivative $y(\Theta_\alpha, t, x)$ by virtue of system (2.1) with a control of the form (2.15). By virtue of the choice of $c_1(t), \ldots, c_r(t)$ we have the equation [7]

$$L(t)B(t) = B_0M(t), \quad (E - B_0B_0^*)\tilde{A}(t) = A_0 \tag{2.16}$$

Then, on the basis of Eq. (2.1) with a control of the form (2.1), using relation (2.15) we obtain

$$\frac{d}{dt}[L(t)x] = A_0L(t)x + \Theta_\alpha^{-1/(2\alpha)}B_0P_0y(\Theta_\alpha, t, x) \tag{2.17}$$

From the relation $y(\Theta_\alpha, t, x) = D(\Theta_\alpha)L(t)x$, using equalities (2.17) and

$$D(\Theta)A_0D^{-1}(\Theta) + D(\Theta)B_0P_0\Theta^{-1/(2\alpha)} = A_1\Theta^{-1/\alpha} \tag{2.18}$$

we have

$$\dot{y}(\Theta_\alpha, t, x) = (\dot{\Theta}_\alpha\Theta_\alpha^{-1}H^\alpha + A_1\Theta_\alpha^{-1/\alpha})y(\Theta_\alpha, t, x) \tag{2.19}$$

Then, from Eq. (2.14), using equalities (2.19) and (2.7), we obtain Eq. (2.13).

It follows from (2.13) that the time of motion $T_\alpha(t_0, x_0)$ from an arbitrary point $x_0 \in Q_\alpha(t_0)$ to the point $x_1 = 0$ is given by the equation

$$T_\alpha(t_0, x_0) = \alpha\Theta_\alpha^{1/\alpha}(t_0, x_0) \tag{2.20}$$

Further, to prove the boundedness of the control and its derivatives, the following result is necessary. We put

$$m_k = \min\{n_0, k\}, \quad \delta_k = \begin{cases} 1 & \text{for} \quad 0 \le k < n_0 \\ 0 & \text{for} \quad n_0 \le k \le l \end{cases}$$

We also put

$$P_k = P_{k-1}((r_k - 1/\alpha)E - H^\alpha + A_1), \quad r_k = k/\alpha + 1/(2\alpha)$$

$$\xi_k(t,\Theta) = \sum_{i=0}^{k} C_k^i\Bigg[[M^{-1}(t)]^{(k-i)}\Theta^{(k-i)/\alpha}P_i - \left(\sum_{j=0}^{k-i} C_{k-i}^j[M^{-1}(t)]^{(k-i-j)}B_0^*\tilde{A}^{(j)}(t)\right)\times \tag{2.21}$$

$$\times\left(\sum_{j=0}^{m_i-1}\Theta^{(k-j)/\alpha}R_{ij} + \delta_i A_0^i D^{-1}(\Theta)\Theta^{r_k}\right)\Bigg]$$

$$R_{ij} = A_0^{m_i-1-j}B_0P_j \tag{2.22}$$

where C_k^i are binomial numbers. Here and everywhere henceforth $k = 0, 1, \dots, l$.

The kth order derivative $(u^\alpha(t,x))^{(k)}$ of the control $u^\alpha(t,x)$, by virtue of the closed system (2.1), is given by the formula

$$(u^\alpha(t,x))^{(k)} = \Theta_\alpha^{-r_k}\xi_k(t,\Theta_\alpha)y(\Theta_\alpha, t, x) \tag{2.23}$$

We will show that the control and its derivatives are bounded. We put

$$\tilde{a}_k = \max_{t_0\le t\le t_1}\left\|B_0^*[\tilde{A}(t)]^{(k)}\right\|, \quad M_k = \max_{t_0\le t\le t_1}\left\|[M^{-1}(t)]^{(k)}\right\|$$

$$\eta_k = \sum_{i=0}^{k} C_k^i\left[c_\alpha^{(k-i)/\alpha}M_{k-i}\|P_i\| + \left(\sum_{j=0}^{k-i}C_{k-i}^jM_{k-i-j}\tilde{a}_j\right)\left(\sum_{j=0}^{m_i-1}c_\alpha^{(k-j)/\alpha}\|R_{ij}\| + \delta_i c_\alpha^{\gamma_i}\right)\right] \tag{2.24}$$

$$\gamma_i = \begin{cases} (k+1)/\alpha, & \text{for} \quad c_\alpha \le 1 \\ (n_0 + k - i)/\alpha, & \text{for} \quad c_\alpha > 1 \end{cases}$$

Here and everywhere henceforth the constant c_α is defined by expression (2.11).

Assertion 4. For each $\alpha \ge 2l + 1$ the control $u^\alpha(t,x)$ and its derivatives $(u^\alpha(t,x))^{(1)}, \dots, (u^\alpha(t,x))^{(l)}$ by virtue of the closed system (2.1), satisfy specified constraints of the form

$$\left\|(u^\alpha(t,x))^{(k)}\right\| \le d_k, \quad x \in Q_\alpha(t)\backslash\{0\}, \quad t \in [t_0, t_0 + T_\alpha) \tag{2.25}$$

Proof. From expression (2.22) we obtain the inequalities

$$\|\xi_k(t, \Theta_\alpha(t,x))\| \le \eta_k, \quad x \in Q_\alpha(t)\backslash\{0\}, \quad t \in [t_0, t_1]$$

Then, from the form of the control (2.15) and its derivatives (2.23), we obtain that when $t \in [t_0, t_0 + T_\alpha) \subset [t_0, t_1]$ the following inequalities hold

$$\left\|(u^\alpha(t,x))^{(k)}\right\| \le \eta_k\|y(\Theta_\alpha, t, x)\|\Theta_\alpha^{-r_k}, \quad x \in Q_\alpha(t)\backslash\{0\} \tag{2.26}$$

From Eq. (2.14) we have

$$\|y(\Theta_\alpha, t, x)\|^2 \le 2a_0\Theta_\alpha\|F_\alpha^{-1}\|, \quad x \in Q_\alpha^1$$

We then obtain from inequalities (2.26)

$$\|(u^\alpha(t,x))^{(k)}\| \le \eta_k\sqrt{2a_0\|F_\alpha^{-1}\|}\,c_\alpha^{1/2-r_k}, \quad x \in Q_\alpha(t)\backslash\{0\}, \quad t \in [t_0, t_0+T_\alpha) \tag{2.27}$$

Choosing a_0 from the condition

$$0 < a_0 \le \min_{0 \le k \le l} d_k^2/(2\|F_\alpha^{-1}\|\eta_k^2 c_\alpha^{1-2r_k}) \tag{2.28}$$

we obtain from inequalities (2.27) that the control and its derivatives satisfy constraints (2.25).

Theorem 1. Consider system (2.1) where $l \ge 1$ is a natural number and rank $B(t) = r$, condition (2.2) is satisfied and we have the expansions (2.4). Suppose $\alpha \ge 2l + 1$, the number a_0 is chosen from condition (2.28), the controllability function $\Theta_\alpha(t, x)$ is defined by Eq. (2.8) and condition (2.9), the constant c_α is defined by expression (2.11), and the set $Q_\alpha(t)$ is defined by expression (2.10).

Then the control $u^\alpha(t, x)$ of the form (2.12) solves the problem of synthesizing inertial controls for system (2.1) for $x \in Q_\alpha(t)\backslash\{0\}$, while the time of motion $T_\alpha(t_0, x_0)$ from an arbitrary point $x(t_0) = x_0 \in Q_\alpha(t_0)$ to the point $x_1 = 0$ is given by Eq. (2.20).

Proof. For each $\alpha \ge 1$ a controllability function $\Theta_\alpha(t, x)$ is constructed which satisfies conditions 1 and 2 (Assertion 1), for which conditions 3 and 5 (Assertion 2) of Theorem 1 from [10] are satisfied. The satisfaction of condition 4 follows from the following. The control $u^\alpha(t,x)$ of the form (2.12) satisfies the Lipschitz condition in each region $\{(t,x)\colon t_0 \le t \le t_1, 0 < \rho_1 \le \|x\| \le \rho_2\}$ with constant $L_u(\rho_1, \rho_2) \to +\infty$ as $\rho_1 \to 0$ and when $\alpha \ge 2l + 1$ together with the derivatives $(u^\alpha(t, x))^{(1)}, \ldots, (u^\alpha(t, x))^{(l)}$ of the form (2.23) satisfies the specified constraints (2.25) (Assertion 4). The derivative of the function $\Theta_\alpha(t, x)$, by virtue of the closed system (2.21) with control (2.12), satisfies Eq. (2.23) (Assertion 3).

Then, we obtain the assertion of this theorem from Theorem 1 from [10].

We will obtain the trajectory $x(t)$ of system (2.1), corresponding to the control $u^\alpha(t, x)$, with begins at an arbitrary point $x_0 \in Q_\alpha(t_0)$ and ends at zero. We choose a_0 from condition (2.28) and find the positive root Θ_α^0 of Eq. (2.8) for $x = x_0$ and $t = t_0$. We consider the Cauchy problem

$$\begin{aligned} \dot{x} &= A(t)x - B(t)M^{-1}(t)B_0^*(1/2F_\alpha(\theta_\alpha(t))L(t) + \dot{L}(t) + L(t)A(t))x \\ x(t_0) &= x_0 \end{aligned} \tag{2.29}$$

$$\dot{\theta}_\alpha(t) = -\theta_\alpha^{1-1/\alpha}(t), \quad \theta_\alpha(t_0) = \Theta_\alpha^0 \tag{2.30}$$

Solving problem (2.30), we have

$$\theta_\alpha(t) = ((t_0 + T_\alpha - t)/\alpha)^\alpha, \quad T_\alpha = \alpha(\Theta_\alpha^0)^{1/\alpha} \tag{2.31}$$

Then, $x(t)$ is the solution of the Cauchy problem corresponding to problem (2.29) after substituting expression (2.31) into the right-hand side of the equation.

We put $z = L(t)x$. Using Eqs (2.16) we obtain

$$\dot{z} = (A_0 - \tfrac{1}{2} B_0 B_0^* F_\alpha(((t_0 + T_\alpha - t)/\alpha)^\alpha))z, \quad z(t_0) = L(t_0)x_0$$

or, in component-by-component form (everywhere henceforth $i = 1, \ldots, r$)

$$\dot{z}_{s_{i-1}+j} = z_{s_{i-1}+j+1}, \quad j = 1, \ldots, n_i - 1, \quad \dot{z}_{s_i} = -\frac{1}{2}\sum_{k=1}^{n_i} \frac{\alpha^{n_i-k+1} f^\alpha_{s_i s_{i-1}+k} z_{s_{i-1}+k}}{(t_0 + T_\alpha - t)^{n_i-k+1}}$$

$$z_{s_{i-1}+j}(t_0) = (\Delta_*^{j-1}c_i(t_0))^* x_0, \quad j = 1, \ldots, n_i$$

where f_{ij}^α are the elements of the matrix F_α. Hence we obtain

$$2(t_0+T_\alpha-t)^{n_i}z^{(n_i)}_{s_{i-1}+1}+\sum_{k=1}^{n_i}\alpha^k f^\alpha_{s_is_i-k+1}(t_0+T_\alpha-t)^{n_i-k}z^{(n_i-k)}_{s_{i-1}+1}=0$$
$$z^{(j)}_{s_{i-1}+1}(t_0)=z_{s_{i-1}+j+1}(t_0),\quad j=0,\ldots,n_i-1 \tag{2.32}$$

We put

$$\Delta_1=-d/d\tau,\quad \Delta_k=(-d/d\tau+k-1)\ldots(-d/d\tau),\quad k=2,\ldots,n_0$$

By replacing the time $t=t_0+T-e^\tau$ from relations (2.32) we have the Cauchy problem in the functions $y_i(\tau)=z_{s_{i-1}+1}(t_0+T_\alpha-e^\tau)$

$$2\Delta_{n_i}y_i(\tau)+\sum_{k=1}^{n_i-1}\alpha^k f^\alpha_{s_is_i-k+1}\Delta_k y_i(\tau)+\alpha^{n_i}f^\alpha_{s_is_{i-1}+1}y_i(\tau)=0$$
$$y_i(\tau_0)=c_i^*(t_0)x_0,\ldots,(\Delta_{n_i-1}y_i)(\tau_0)=T_\alpha^{n_i-1}(\Delta_*^{n_i-1}c_i(t_0))^*x_0;\quad \tau_0=\ln(t_0+T_\alpha)$$

Since

$$z_{s_{i-1}+1}(t)=y_i(\ln(t_0+T_\alpha-t))$$

the remaining functions $z_{s_{i-1}+2}(t),\ldots,z_{s_i}(t)$ are found by differentiating the last equation, i.e.

$$z_{s_{i-1}+j}(t)=z^{(j-1)}_{s_{i-1}+1}(t),\quad j=2,\ldots,n_i$$

The trajectory $x(t)$ is defined by the equality $x(t)=L^{-1}(t)z(t)$ and, as can be seen from the above discussion, one only needs to solve Eq. (2.8) once to find it.

Example. Consider the problem of the positional synthesis of inertial controls for a model two-dimensional system of the form

$$\dot x_1=\frac{1}{1+t}x_1+\frac{1}{(1+t)^2}x_2+\frac{1}{1+t}u$$
$$\dot x_2=-x_1-\frac{2}{1+t}x_2+u,\quad t\in[0,3] \tag{2.33}$$

with constraints on the control and its derivative of the form (1.3), where $d_0=1$ and $d_1=3$. System (2.33) is completely controllable since condition (2.2) is satisfied for $t\geq 0$. We will consider the case when $\alpha=3$, and this subscript will not be indicated in the notation. We will choose a number a_0 from condition (2.28), putting it equal to $6/(136+43\sqrt{10})$. The equation for determining the function $\Theta(t,x)$ when $x\neq 0$, according to Eq. (2.8), has the form

$$\frac{12}{136+43\sqrt{10}}\Theta^2-\frac{10}{27}\Theta^{2/3}(2(1+t)x_1+x_2)^2-\frac{10}{27}\Theta^{1/3}(2(1+t)x_1+x_2)\times$$
$$\times((1+t)^2x_1-(1+t)x_2)-\frac{25}{162}(1+t)^2((1+t)x_1-x_2)^2=0 \tag{2.34}$$

From condition (2.11) we obtain that, when $\bar\Theta\geq 8123956$ and values of δ and σ close to unity, the constant $c=1$. The region $Q(t)$ has the form

$$Q(t)=\{(x_1,x_2):(1+t)^2(77+34t+5t^2)x_1^2-2(1+t)(-13+16t+$$
$$+5t^2)x_1x_2+(5-2t+5t^2)x_2^2\leq 1944/(680+215\sqrt{10})\},\quad t\in[0,3]$$

The control $u(t,x)$ from (2.12) is given by the formula

$$u(t,x) = -\left(\frac{5}{3\Theta^{2/3}(t,x)} + \frac{2}{(1+t)^2}\right)\frac{1+t}{6}((1+t)x_1 - x_2) -$$
$$-\left(\frac{5}{3\Theta^{1/3}(t,x)} + \frac{1}{1+t}\right)\left(\frac{2(1+t)}{3}x_1 + \frac{1}{3}x_2\right)$$

This control solves the problem of synthesizing inertial controls for system (2.33) in the region $Q(t)\backslash\{0\}$, $t \in [0, 3]$, and together with the derivative

$$\dot{u}(t,x) = \left(\frac{5}{3\Theta(t,x)} + \frac{5}{3(1+t)\Theta^{2/3}(t,x)} + \frac{4}{(1+t)^3}\right)\frac{1+t}{6}((1+t)x_1 - x_2) +$$
$$+\left(\frac{5}{9\Theta^{2/3}(t,x)} + \frac{5}{3(1+t)\Theta^{1/3}(t,x)} - \frac{1}{(1+t)^2}\right)\left(\frac{2(1+t)}{3}x_1 + \frac{1}{3}x_2\right)$$

satisfies the constraints $|u(t,x)| \le 1$, $|\dot{u}(t,x)| \le 3$ in it.

Suppose Θ^0 is the positive root of Eq. (2.34) when $t = 0$ and $x = x_0$. We will introduce the notation

$$T = 3(\Theta^0)^{1/3}, \quad \gamma(t) = \sqrt{6}\ln(T-t), \quad \gamma_0 = \sqrt{6}\ln T$$

$$\begin{Bmatrix} k_1 \\ k_2 \end{Bmatrix} = T^{-3}\left(\frac{1}{6}(x_1^0 - x_2^0)\left(\begin{Bmatrix} \cos \\ \sin \end{Bmatrix}\gamma_0 + \frac{3}{\sqrt{6}}\begin{Bmatrix} \sin \\ -\cos \end{Bmatrix}\gamma_0\right) + \frac{1}{3\sqrt{6}}(2x_1^0 - x_2^0)T\begin{Bmatrix} \sin \\ -\cos \end{Bmatrix}\gamma_0\right)$$

The trajectory of system (2.33) corresponding to the control $u(t, x)$ and proceeding from the point $x(0) = x_0 \in Q(0)$ to zero, is given by the equations

$$x_1(t) = \frac{2}{(1+t)^2}z_1(t) + \frac{1}{1+t}z_2(t), \quad x_2(t) = -\frac{4}{1+t}z_1(t) + z_2(t)$$

$$z_1(t) = (T-t)^3(k_1\cos\gamma(t) + k_2\sin\gamma(t))$$

$$z_2(t) = (T-t)^2(-(3k_1 + \sqrt{6}k_2)\cos\gamma(t) + (\sqrt{6}k_1 - 3k_2)\sin\gamma(t))$$

The control and its derivative along this trajectory have the form

$$u(t) = -\left(\frac{15}{(T-t)^2} + \frac{2}{(1+t)^2}\right)z_1(t) - \left(\frac{5}{T-t} + \frac{1}{1+t}\right)z_2(t)$$

$$\dot{u}(t) = \left(\frac{45}{(T-t)^3} + \frac{15}{(1+t)(T-t)^2} + \frac{4}{(1+t)^3}\right)z_1(t) + \left(\frac{5}{2(T-t)^2} + \frac{5}{(1+t)(T-t)} - \frac{1}{(1+t)^2}\right)z_2(t)$$

The region $Q(0)$ (its boundary is represented by the thick curve) and the phase trajectories, which transfer the points (0.1, 0.1), (–0.17, 0.5), (0.13, –0.67) $\in Q(0)$ to zero, after a time $T \approx 2.797$, $T \approx 2.944$ and $T \approx 2.945$ respectively, are shown in Fig. 1. In Fig. 2 we show the control and its derivative on trajectories which begin at the point (0.13, –0.67) $\in Q(0)$ and end at zero. They obviously satisfy the specified constraints.

3. SYNTHESIS OF CONTROLS FOR A NON-LINEAR SYSTEM TO A FIRST APPROXIMATION

We will consider the problem of synthesizing controls for system (1.1) with constraints on the control of the form (1.3) when $l = 1$. We will assume that the function $f(t, x, u)$ satisfies the condition $f(t, 0, 0) = 0$ (everywhere henceforth we must again bear in mind that $t \in [t_0, t_1]$) and has derivatives with respect to x and u that are continuous up to the second order. Then, in the neighbourhood of zero, we can write system (1.1) in the form

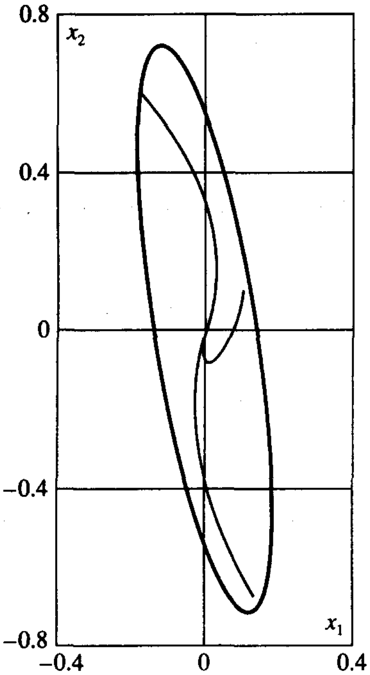

Fig. 1

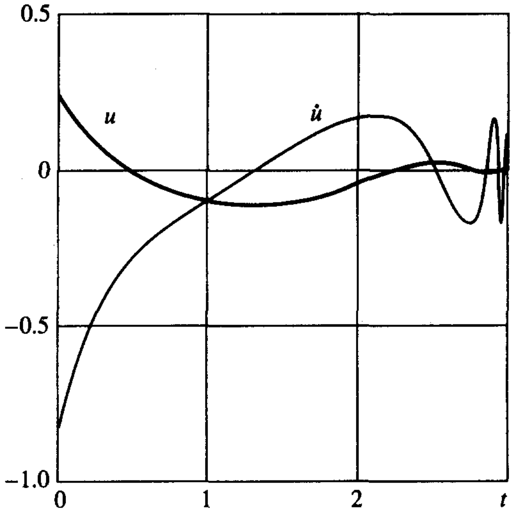

Fig. 2

$$\dot{x} = A(t)x + B(t)u + g(t, x, u); \quad A(t) = f_x(t, 0, 0), \quad B(t) = f_u(t, 0, 0) \tag{3.1}$$

Here $g(t, x, u)$ is a continuous function; we will assume that it satisfies the inequality

$$\|g(t, x, u)\| \le c_1\|x\|^{s_1} + c_2\|x\|^{s_2}\|u\|^{s_3} + c_3\|u\|^{s_4} \tag{3.2}$$

where

$$c_1 \ge 0, \quad c_2 \ge 0, \quad c_3 \ge 0, \quad s_1 > 1, \quad s_2 + s_3 > 1, \quad s_4 > 1$$

We will put

$$\alpha_0 = \max\left\{3, \frac{2n_0 - 4}{s_1 - 1} - 1, \frac{2n_0 + s_3 - s_2 - 3}{s_2 + s_3 - 1}, \frac{2n_0 - 2}{s_4 - 1} - 1, \frac{2n_0}{s_1} - 1, \frac{2n_0 - s_2 + s_3}{s_2 + s_3}, \frac{2n_0}{s_4} + 1\right\}$$

Theorem 2. We will consider the controllable system (3.1) where $A(t) \in C^{2n-1}$, $B(t) \in C^{2n}$, $\operatorname{rank}B(t) = r$, for which condition (2.3) is satisfied, and we have expansions (2.4). Suppose the function $g(t, x, u)$ satisfies inequality (3.2) and in each region

$$\{(t, x, u) : t_0 \le t \le t_1, 0 < \rho_1 \le \|x\| \le \rho_2, \|u\| \le d_0\}$$

satisfies the Lipschitz condition

$$\|g(t, x'', u'') - g(t, x', u')\| \le L_g(\rho_1, \rho_2)(\|x'' - x'\| + \|u'' - u'\|)$$

Then, positive numbers a_0 and $\tilde{c}_\alpha < 1$ exist such that, when $\alpha \ge \alpha_0$, the control

$$u^\alpha(t, x) = -1/2M^{-1}(t)B_0^* F_\alpha(\Theta_\alpha(t, x))L(t)x \tag{3.3}$$

where the controllability function $\Theta_\alpha(t, x)$ is defined by Eq. (2.8) and equality (2.9), solves the problem of synthesizing controls for system (3.1) in the region $\tilde{Q}_\alpha(t) = \{x: \Theta_\alpha(t, x) \le \tilde{c}_\alpha\}$ and satisfies the constraints

$$\|u^\alpha(t, x)\| \le d_0, \quad \|\dot{u}^\alpha(t, x)\| \le d_1 \tag{3.4}$$

The time of motion $T_\alpha(t_0, x_0)$ from the point $x(t_0) = x_0 \in \tilde{Q}_\alpha(t_0)$ to the point 0 along the trajectory of system (3.1) with control $u^\alpha(t, x)$ satisfies the inequality

$$T_\alpha(t_0, x_0) \le (\alpha/\tilde{\beta}_\alpha)\Theta_\alpha^{1/\alpha}(t_0, x_0), \quad \tilde{\beta}_\alpha > 0$$

Proof. Taking into account the result proved in Section 2, by Theorem 1 from [10] for the complete proof of this theorem we need to show that the control and its derivative, by virtue of the closed system (3.1), satisfy the specified constraints, and we need to establish inequality (1.4) for system (3.1) with control (3.3), the satisfaction of which ensures that the trajectory will be incident on the origin of coordinates after a finite time.

We will put

$$y(\Theta, t, x) = D(\Theta)L(t)x$$

and rewrite control (3.3) in the form

$$u^\alpha(t, x) = M^{-1}(t)\Theta_\alpha^{-1/(2\alpha)}(t, x)P_0 y(\Theta_\alpha(t, x), t, x) \tag{3.5}$$

We will further assume that

$$\Theta = \Theta_\alpha(t, x), \quad y = y(\Theta_\alpha(t, x), t, x), \quad D = D(\Theta_\alpha(t, x)), \quad g = g(t, x, u^\alpha(t, x))$$

On the basis of Eq. (3.1) with control (3.5) and Eqs (2.16) we have

$$d(L(t)x)/dt = A_0 L(t)x + \Theta^{-1/(2\alpha)}B_0 P_0 y + L(t)g$$

Then, as above, using Eq. (2.18), we obtain

$$\dot{y} = (\dot{\Theta}\Theta^{-1}H^\alpha + \Theta^{-1/\alpha}A_1 + \Theta^{-1/(2\alpha)}B_0 B_0^* \tilde{A}(t)D^{-1})y + DL(t)g \tag{3.6}$$

From Eq. (2.14), using relations (3.6), (2.14) and (2.7), we have

$$\begin{aligned} \dot{\Theta} &= -\Theta^{1-1/\alpha} + (\Theta^{1-1/(2\alpha)}(\chi(t)y, y) + 2\Theta(F_\alpha y, DL(t)g))/(F^\alpha y, y) \\ \chi(t) &= F_\alpha B_0 B_0^* \tilde{A}(t)D^{-1} + D^{-1}\tilde{A}^*(t)B_0 B_0^* F_\alpha \end{aligned} \tag{3.7}$$

Then, on the basis of Eq. (3.7), the derivative of the control $u^\alpha(t, x)$ of the form (3.5), by virtue of the closed system (3.1), has the form

$$\begin{aligned} \dot{u}^\alpha(t, x) &= M_t^{-1}(t)\Theta^{-1/(2\alpha)}P_0 y + M^{-1}(t)\Theta^{-3/(2\alpha)}P_1 y + M^{-1}(t)\Theta^{-1/(2\alpha)} \times \\ &\times P_0 DL(t)g + M^{-1}(t)\Theta^{-1/\alpha}P_0 B_0 B_0^* \tilde{A}(t)D^{-1}y + M^{-1}(t)P_0(H^\alpha - E/(2\alpha))y \times \\ &\times [\Theta^{-1/\alpha}(\chi(t)y, y) + 2\Theta^{-1/\alpha}(F_\alpha y, DL(t)g)]/(F^\alpha y, y) \end{aligned} \tag{3.8}$$

From (2.14) we have

$$\sqrt{2a_0\Theta/\|F_\alpha\|} \le \|y\| \le \sqrt{2a_0\Theta\|F_\alpha^{-1}\|} \tag{3.9}$$

Then, since

$$(F^\alpha y, y) \ge \|y\|^2/\|(F^\alpha)^{-1}\|$$

$$\|D(\Theta)\| \le \Theta^{-n_0/\alpha + 1/(2\alpha)}, \quad \|D^{-1}(\Theta)\| \le \Theta^{1/(2\alpha)}, \quad \Theta \le 1$$

from relations (3.7), (3.5) and (3.8) we obtain the inequalities

$$\begin{aligned} \dot{\Theta} &\le -(1 - 2\Theta^{1/\alpha}\|F_\alpha\|\|(F^\alpha)^{-1}\|\tilde{a}_0 - \\ &- \sqrt{2/a_0}\Theta^{-1/2 - n_0/\alpha + 3/(2\alpha)}L_{\max}\|F_\alpha\|^{3/2}\|(F^\alpha)^{-1}\|\|g\|)\Theta^{1-1/\alpha} \end{aligned} \tag{3.10}$$

$$\|u^{\alpha}(t,x)\| \le \mu_0\sqrt{a_0}\Theta^{1/2-1/(2\alpha)}; \quad \mu_0 = M_0\|F_\alpha\|\|F_\alpha^{-1}\|^{1/2}/\sqrt{2} \tag{3.11}$$

$$\begin{aligned}\|\dot{u}^{\alpha}(t,x)\| &\ge \mu_1\sqrt{a_0}\Theta^{1/2-3/(2\alpha)} + \mu_2\sqrt{a_0}\Theta^{1/2-1/(2\alpha)} + \mu_3\Theta^{-n_0/\alpha}\|g\|\\ \mu_1 &= \mu_0(1+n_0/\alpha+\|F_\alpha\|/2)\\ \mu_2 &= \|F_\alpha\|\|F_\alpha^{-1}\|^{1/2}(M_1+M_0\tilde{a}_0+2n_0\|F_\alpha\|\|(F^{\alpha})^{-1}\|\tilde{a}_0/\alpha)/\sqrt{2}\\ \mu_3 &= M_0\|F_\alpha\|(1/2+n_0\|F_\alpha\|\|(F^{\alpha})^{-1}\|/\alpha)L_{\max}\end{aligned} \tag{3.12}$$

The quantities $\tilde{a}_0$, M_0, M_1 are defined by formulae (2.24).

We will obtain an estimate for $\|g(L^{-1}(t)D^{-1}y, u^{\alpha}(t,x))\|$. Using inequality (3.2), the form of the control $u^{\alpha}(t,x)$ and the right-hand side of inequality (3.9), we have

$$\begin{aligned}\|g\| &\le \mu_4 a_0^{s_1/2}\Theta^{s_1/(2\alpha)+s_1/2} + \mu_5 a_0^{(s_2+s_3)/2}\Theta^{(s_2+s_3)/2+(s_2-s_3)/(2\alpha)} + \mu_6 a_0^{s_4/2}\Theta^{s_4/2-s_4/(2\alpha)}\\ \mu_4 &= c_1 2^{s_1/2}L_0^{s_1}\|F_\alpha^{-1}\|^{s_1/2}\\ \mu_5 &= c_2 2^{(s_2-s_3)/2}L_0^{s_2}M_0^{s_3}\|F_\alpha\|^{s_3}\|F_\alpha^{-1}\|^{(s_2+s_3)/2}\\ \mu_6 &= c_3 2^{-s_4/2}M_0^{s_4}\|F_\alpha\|^{s_4}\|F_\alpha^{-1}\|^{s_4/2}\end{aligned} \tag{3.13}$$

We then obtain the following inequality from (3.10)

$$\dot{\Theta} \le -\beta_\alpha(\Theta)\Theta^{1-1/\alpha} \tag{3.14}$$

where

$$\begin{aligned}\beta_\alpha(\Theta) &\doteq 1 - 2\Theta^{1/\alpha}\|F_\alpha\|\|(F^{\alpha})^{-1}\|\tilde{a}_0 - \sqrt{2}\|F_\alpha\|^{3/2}\|(F^{\alpha})^{-1}\|L_{\max}\times\\ &\times(\mu_4 a_0^{(s_1-1)/2}\Theta^{\nu_1(\alpha)} + \mu_5 a_0^{(s_2+s_3-1)/2}\Theta^{\nu_2(\alpha)} + \mu_6 a_0^{(s_4-1)/2}\Theta^{\nu_3(\alpha)})\\ \nu_1(\alpha) &= (s_1-1)/2 - n_0/\alpha + (s_1+3)/(2\alpha) \ge 0\\ \nu_2(\alpha) &= (s_2+s_3-1)/2 - n_0/\alpha + (s_2-s_3+3)/(2\alpha) \ge 0\\ \nu_3(\alpha) &= (s_4-1)/2 - n_0/\alpha + (3-s_4)/(2\alpha) \ge 0\end{aligned}$$

when $\alpha \ge \alpha_0$. From inequalities (3.13), (3.11) and (3.12) we obtain

$$\|u^{\alpha}(t,x)\| \le \mu_0\sqrt{a_0}, \quad \|\dot{u}^{\alpha}(t,x)\| \le \psi(a_0), \quad x \in \{x : \Theta_\alpha(t,x) \le \min\{c_\alpha, 1\}\} \tag{3.15}$$

where

$$\psi(a_0) = (\mu_1+\mu_2)\sqrt{a_0} + \mu_3(\mu_4 a_0^{s_1/2} + \mu_5 a_0^{(s_2+s_3)/2} + \mu_6 a_0^{s_4/2})$$

Suppose the number a_0 satisfies the inequalities

$$0 < a_0 \le d_0^2/\mu_0^2, \quad \psi(a_0) \le d_1$$

We choose a positive constant $\hat{c}_\alpha$ such that for $0 < \Theta \le \hat{c}_\alpha$ the inequality $\beta_\alpha(\Theta) > 0$ is satisfied. We choose $\tilde{c}_\alpha = \min\{c_\alpha, \hat{c}_\alpha, 1\}$, and consequently, we have $\tilde{Q}_\alpha(t) \subset Q_\alpha(t) \subset Q_\alpha^1$. For these a_0 and $\tilde{c}_\alpha$ we put $\tilde{\beta}_\alpha = \beta_\alpha(\tilde{c}_\alpha)$. From (3.14) we then obtain the inequality

$$\dot{\Theta}_\alpha(t,x) \le -\tilde{\beta}_\alpha\Theta_\alpha^{1-1/\alpha}(t,x), \quad x \in \tilde{Q}_\alpha(t)$$

Hence, we have established inequality (1.4) when $\beta = \tilde{\beta}_\alpha$ and $\alpha \ge \alpha_0$.

It follows from inequalities (3.15) that the control and its derivative satisfy constraints (3.4) for $x \in \tilde{Q}_{\alpha}(t)\backslash\{0\}$. The assertion of Theorem 2 follows from Theorem 1 of [10].

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